C4 JUNE 2010

1.

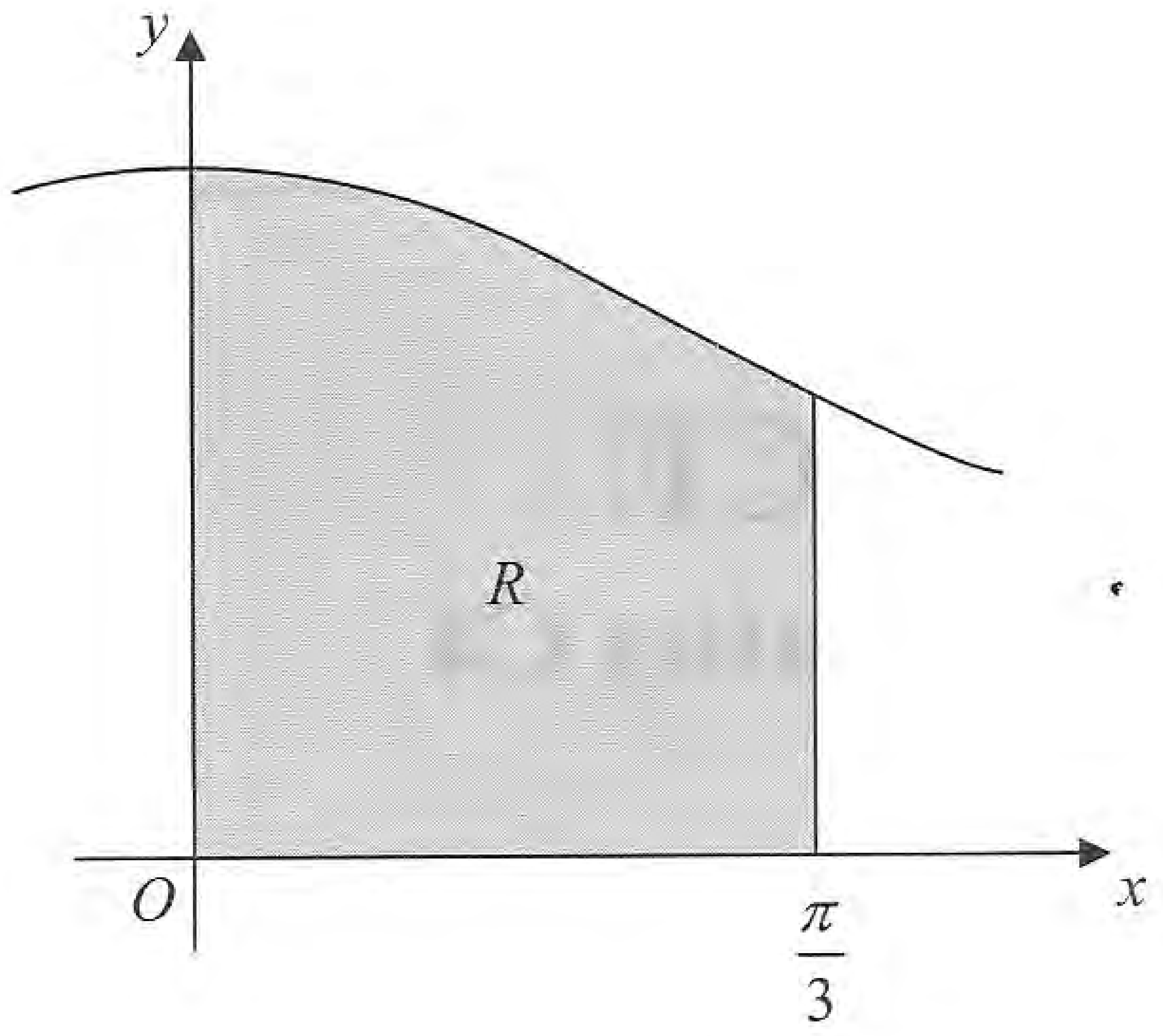


Figure 1

Figure 1 shows part of the curve with equation $y = \sqrt{(0.75 + \cos^2 x)}$. The finite region R, shown shaded in Figure 1, is bounded by the curve, the y-axis, the x-axis and the line with equation $x = \frac{\pi}{3}$.

(a) Complete the table with values of y corresponding to $x = \frac{\pi}{6}$ and $x = \frac{\pi}{4}$.

x	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
y	1.3229	1.2973	1.2247	1.1180	1

(2)

(b) Use the trapezium rule

(i) with the values of y at $x = 0$, $x = \frac{\pi}{6}$ and $x = \frac{\pi}{3}$ to find an estimate of the area of R. Give your answer to 3 decimal places.

(ii) with the values of y at $x = 0$, $x = \frac{\pi}{12}$, $x = \frac{\pi}{6}$, $x = \frac{\pi}{4}$ and $x = \frac{\pi}{3}$ to find a further estimate of the area of R. Give your answer to 3 decimal places.

(6)

i) $\frac{1}{2}\left(\frac{\pi}{6}\right)\left(1.3229 + 2(1.2247) + 1\right) = \underline{1.249}$ (3dp)

ii) $\frac{1}{2}\left(\frac{\pi}{12}\right)\left(1.3229 + 2(1.2973 + 1.2247 + 1.118) + 1\right) = \underline{1.257}$ (3dp)

2. Using the substitution $u = \cos x + 1$, or otherwise, show that

$$\int_0^{\frac{\pi}{2}} e^{\cos x + 1} \sin x \, dx = e(e-1)$$

(6)

$u = \cos x + 1 \Rightarrow \frac{du}{dx} = -\sin x \Rightarrow dx = \frac{-1}{\sin x} du$

$x = \frac{\pi}{2} \quad u = \cos\left(\frac{\pi}{2}\right) + 1 = 1$

$x = 0 \quad u = \cos(0) + 1 = 2$

$\Rightarrow \int_2^1 e^u \cancel{\sin x} \times \frac{-1}{\cancel{\sin x}} du = -\int_2^1 e^u du = +\int_1^2 e^u du$

$= \left[e^u\right]_1^2 = e^2 - e^1 = (e^1)^2 - e^1 = e^1(e^1 - 1)$

$= \underline{e(e-1)}$ qed

3. A curve C has equation

$$2^x + y^2 = 2xy$$

Find the exact value of $\frac{dy}{dx}$ at the point on C with coordinates $(3, 2)$.

(7)

$$\frac{d}{dx}2^x + \frac{d}{dx}y^2 = \frac{d}{dx}2xy$$

$$\frac{d}{dx}2^x \qquad u = 2^x \Rightarrow \ln u = \ln 2^x \Rightarrow \ln u = x\ln 2$$

$$\frac{d}{dx}\ln u = \frac{d}{dx}x\ln 2 \Rightarrow \frac{1}{u}\frac{du}{dx} = \ln 2$$

$$\frac{du}{dx} = u\ln 2 \Rightarrow \frac{du}{dx} = 2^x \ln 2$$

$$\frac{d}{dx}y^2 = 2y\frac{dy}{dx} \qquad \frac{d}{dx}2xy \qquad \begin{array}{ll} u = 2x & v = y \\ u' = 2 & \frac{dv}{dx} = \frac{dy}{dx} \end{array}$$

$$\frac{d}{dx}2xy = 2y + 2x\frac{dy}{dx}$$

$$\Rightarrow 2^x\ln 2 + 2y\frac{dy}{dx} = 2y + 2x\frac{dy}{dx}$$

$$-2y + 2^x\ln 2 = (2x - 2y)\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2y + 2^x\ln 2}{2x - 2y}$$

$$(3,2) \Rightarrow \frac{dy}{dx} = \frac{-4 + 8\ln 2}{6 - 4} = \frac{-4 + 8\ln 2}{2} = \underline{-2 + 4\ln 2}$$

4. A curve C has parametric equations

$$x = \sin^2 t, \quad y = 2\tan t, \quad 0 \leqslant t < \frac{\pi}{2}$$

(a) Find $\frac{dy}{dx}$ in terms of t.

(4)

The tangent to C at the point where $t = \frac{\pi}{3}$ cuts the x-axis at the point P.

(b) Find the x-coordinate of P.

(6)

$\frac{dx}{dt} = 2\sin t\cos t \qquad \frac{dy}{dt} = 2\sec^2 t = \frac{2}{\cos^2 t}$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{\cancel{2}}{\cos^2 t} \times \frac{1}{\cancel{2}\sin t\cos t} = \frac{1}{\sin t\cos^3 t}$

b) $t = \frac{\pi}{3} \quad x = \sin^2\left(\frac{\pi}{3}\right) = \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}$

$y = 2\tan\left(\frac{\pi}{3}\right) = 2\sqrt{3}$

m_t at $t = \frac{\pi}{3} = \frac{1}{\sin\left(\frac{\pi}{3}\right)\cos^3\left(\frac{\pi}{3}\right)} = \frac{1}{\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)^3} = \frac{16}{\sqrt{3}}$

$y - 2\sqrt{3} = \frac{16}{\sqrt{3}}\left(x - \frac{3}{4}\right) \Rightarrow \sqrt{3}y - 6 = 16x - 12$

$y\sqrt{3} = 16x - 6$

Cuts x-axis when $y = 0 \Rightarrow 16x = 6 \quad x = \frac{6}{16} = \frac{3}{8}$

5. $$\frac{2x^2+5x-10}{(x-1)(x+2)} \equiv A + \frac{B}{x-1} + \frac{C}{x+2}$$

(a) Find the values of the constants A, B and C.

(4)

(b) Hence, or otherwise, expand $\dfrac{2x^2+5x-10}{(x-1)(x+2)}$ in ascending powers of x, as far as the term in x^2. Give each coefficient as a simplified fraction.

(7)

$2x^2+5x-10 = A(x-1)(x+2) + B(x+2) + C(x-1)$

$x=1 \Rightarrow -3 = 3B \quad \underline{B=-1}$

$x=-2 \Rightarrow -12 = -3C \quad \underline{C=4}$

$x=0 \Rightarrow -10 = -2A+2B-C \Rightarrow -4 = -2A \Rightarrow \underline{A=2}$

b) $\dfrac{2x^2+5x-10}{(x-1)(x+2)} = 2 + \dfrac{-1}{(x-1)} + \dfrac{4}{x+2} = 2 + \dfrac{1}{1-x} + \dfrac{4}{2+x}$

$= 2 + (1-x)^{-1} + 4(2+x)^{-1} = 2 + (1-x)^{-1} + 4\times2^{-1}\left(1+\frac{x}{2}\right)^{-1}$

$= 2 + (1-x)^{-1} + 2\left(1+\frac{x}{2}\right)^{-1}$

$= 2 + \left(1 + (-1)(-x) + \frac{(-1)(-2)}{2}(-x)^2\right) + 2\left(1 + (-1)\left(\frac{x}{2}\right) + \frac{(-1)(-2)}{2}\left(\frac{x}{2}\right)^2\right)$

$= 2 + (1 + x + x^2) + (2 + x + \frac{1}{2}x^2)$

$= \quad 5 + \frac{3}{2}x^2$

6.

$$f(\theta) = 4\cos^2\theta - 3\sin^2\theta$$

(a) Show that $f(\theta) = \frac{1}{2} + \frac{7}{2}\cos 2\theta$. **(3)**

(b) Hence, using calculus, find the exact value of $\int_0^{\frac{\pi}{2}} \theta f(\theta)\, d\theta$. **(7)**

$\cos 2\theta = 2\cos^2\theta - 1 \Rightarrow 2\cos^2\theta = \cos 2\theta + 1 \Rightarrow 4\cos^2\theta = 2\cos 2\theta + 2$

$\cos 2\theta = 1 - 2\sin^2\theta \Rightarrow 2\sin^2\theta = 1 - \cos 2\theta \Rightarrow 3\sin^2\theta = \frac{3}{2} - \frac{3}{2}\cos 2\theta$

$\Rightarrow 4\cos^2\theta - 3\sin^2\theta = 2\cos 2\theta + 2 - \frac{3}{2} + \frac{3}{2}\cos 2\theta$

$= \frac{1}{2} + \frac{7}{2}\cos 2\theta$ qed.

b) $\int_0^{\frac{\pi}{2}} \theta f(\theta) d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2}\theta\, d\theta + \int_0^{\frac{\pi}{2}} \theta \frac{7}{2}\cos 2\theta\, d\theta$

$u = \frac{7}{2}\theta \quad v = \frac{1}{2}\sin 2\theta$

$u' = \frac{7}{2} \quad v' = \cos 2\theta$

$= \left[\frac{1}{4}\theta^2 + \frac{7}{4}\theta\sin 2\theta - \int \frac{7}{4}\sin 2\theta\, d\theta \right]_0^{\frac{\pi}{2}}$

$= \left[\frac{1}{4}\theta^2 + \frac{7}{4}\theta\sin 2\theta + \frac{7}{8}\cos 2\theta \right]_0^{\frac{\pi}{2}}$

$= \left(\frac{1}{4}\left(\frac{\pi}{2}\right)^2 + 0 - \frac{7}{8} \right) - \left(0 + 0 + \frac{7}{8} \right) = \frac{\pi^2}{16} - \frac{7}{4} = \frac{1}{16}(\pi^2 - 28)$

7. The line l_1 has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, where λ is a scalar parameter.

The line l_2 has equation $\mathbf{r} = \begin{pmatrix} 0 \\ 9 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}$, where μ is a scalar parameter.

Given that l_1 and l_2 meet at the point C, find

(a) the coordinates of C.

(3)

The point A is the point on l_1 where $\lambda = 0$ and the point B is the point on l_2 where $\mu = -1$.

(b) Find the size of the angle ACB. Give your answer in degrees to 2 decimal places.

(4)

(c) Hence, or otherwise, find the area of the triangle ABC.

(5)

$l_1 = l_2 \Rightarrow \begin{pmatrix} 2+\lambda \\ 3+2\lambda \\ -4+\lambda \end{pmatrix} = \begin{pmatrix} 5\mu \\ 9 \\ -3+2\mu \end{pmatrix}$ j) $2\lambda = 6 \Rightarrow \lambda = 3$

i) $5 = 5\mu \Rightarrow \mu = 1$

$\lambda = 3 \Rightarrow C(5, 9, -1)$

b) $\lambda = 0 \Rightarrow A(2, 3, -4)$ $\mu = -1 \Rightarrow B(-5, 9, -5)$

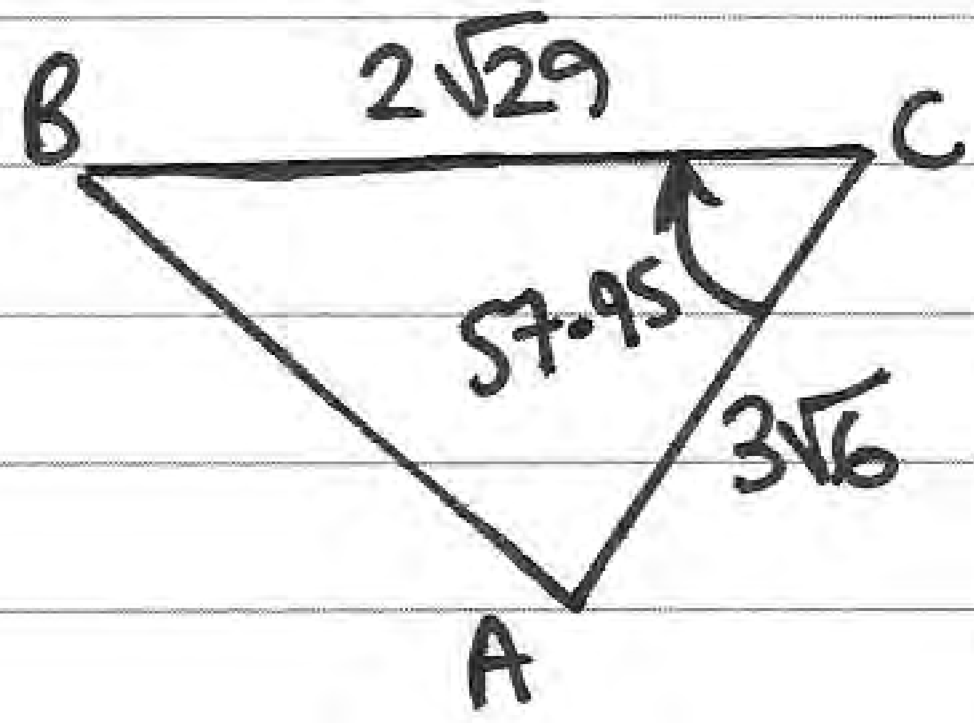
$\vec{CA} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} - \begin{pmatrix} 5 \\ 9 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ -6 \\ -3 \end{pmatrix}$ $|\vec{CA}| = \sqrt{3^2+6^2+3^2} = 3\sqrt{6}$

$\vec{CB} = \begin{pmatrix} -5 \\ 9 \\ -5 \end{pmatrix} - \begin{pmatrix} 5 \\ 9 \\ -1 \end{pmatrix} = \begin{pmatrix} -10 \\ 0 \\ -4 \end{pmatrix}$ $|\vec{CB}| = \sqrt{10^2+4^2} = 2\sqrt{29}$

$\vec{CA}.\vec{CB} = 30 + 0 + 12 = 42$

$\cos\theta = \dfrac{\vec{CA}.\vec{CB}}{|\vec{CA}||\vec{CB}|} = \dfrac{42}{6\sqrt{174}} \Rightarrow \theta = 57.95°$

c)



$\text{Area} = \frac{1}{2}(2\sqrt{29})(3\sqrt{6})\sin 57.95...$

$\text{Area} = \underline{33.54}$

8.

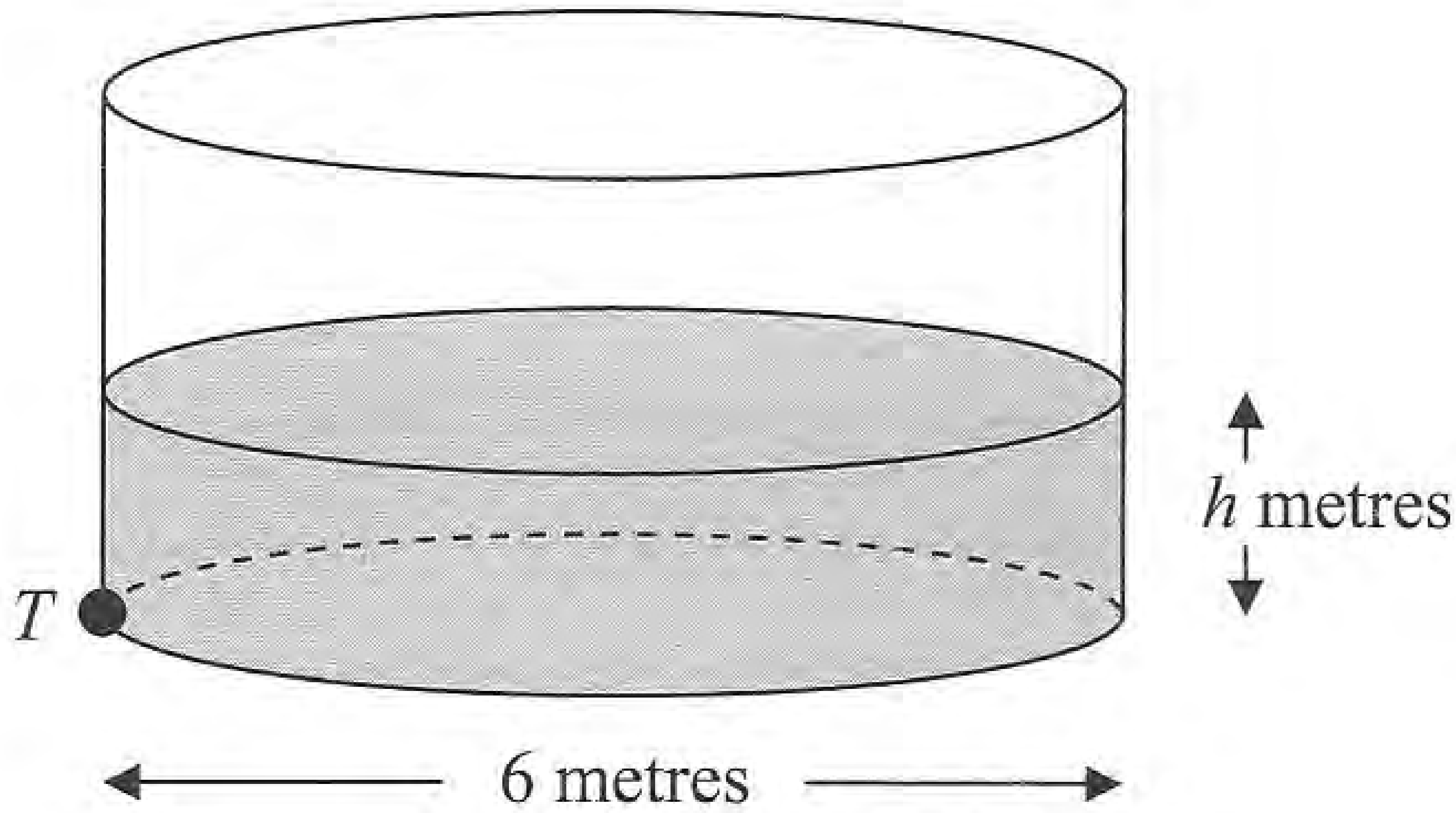


Figure 2

Figure 2 shows a cylindrical water tank. The diameter of a circular cross-section of the tank is 6 m. Water is flowing into the tank at a constant rate of 0.48π m^3 min^{-1}. At time t minutes, the depth of the water in the tank is h metres. There is a tap at a point T at the bottom of the tank. When the tap is open, water leaves the tank at a rate of $0.6\pi h$ m^3 min^{-1}.

(a) Show that t minutes after the tap has been opened

$$75\frac{\mathrm{d}h}{\mathrm{d}t} = (4 - 5h)$$

(5)

When $t = 0$, $h = 0.2$

(b) Find the value of t when $h = 0.5$

(6)

a) $\frac{dV}{dt} = 0.48\pi - 0.6\pi h$ $\qquad V = \pi r^2 h = 9\pi h$

$\frac{dV}{dh} = 9\pi \Rightarrow \frac{dh}{dV} = \frac{1}{9\pi}$

$\frac{dh}{dt} = \frac{dh}{dV}\frac{dV}{dt} = \frac{1}{9\pi}(0.48\pi - 0.6\pi h)$

$\frac{dh}{dt} = \frac{4}{75} - \frac{1}{15}h \Rightarrow \frac{dh}{dt} = \frac{1}{75}(4-5h) \Rightarrow 75\frac{dh}{dt} = 4-5h$ qed.

b) $-15\int \frac{-5}{4-5h}\,dh = \int 1\,dt \Rightarrow -15\ln(4-5h) = t + C$

$t=0, h=0.2 \Rightarrow -15\ln 3 = C$

$t = 15\ln 3 - 15\ln(4-5h) \Rightarrow t = 15\ln\left(\frac{3}{4-5h}\right)$

$h = 0.5 \Rightarrow t = 15\ln\frac{3}{1.5} = 15\ln 2$ $\qquad \underline{t = 10.4}$ (3sf)